V P N Nampoori

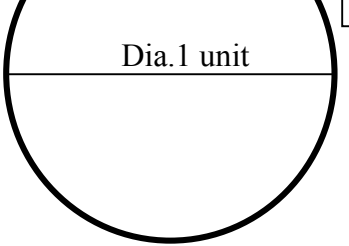
V P N Nampoori
International School of Photonics
Cochin University of Science and Technology

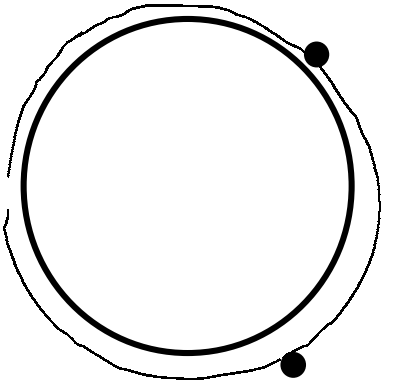
Measuring the immeasurable

The Strange Story of π

Pi is among the few numbers which have revolutionized human knowledge . Wide range of human activity in the field of science and technology owes their progress to this enigmatic number. So real yet immeasurable. Draw a circle of diameter I unit. The circumference of the circle is Pi- it is there in flesh and blood.

Circumference of this circle is 1 Pi units





Circumference of the circle is slightly more than three times the diameter

Now try to measure it using a tape . Take a thread superpose over the circle so that it exactly overlays along the circumference of the circle. Take out the thread. Using a metre scale, which has least count of only 1cm, measure the length of the thread. The length will be slightly more than three 3cm. Use another scale with length which can measure up to a mm. Then the circumference will be slightly more than 3.1cm. With one micron scale, length will be slightly more than 3.1415 cm. With 1 nm scale it will measure more than 3.1415926 cm. and with picometre scale more

completely.

The whole story started by the one of the most remarkable discoveries of human race – the wheel. Imagine the fate of human race if wheel was not invented! On every sphere of human activity this wheel has a stamp.

Invention of the wheel brought another discovery: *The circumference increases with diameter of the wheel.* It was also found that only three components are needed to describe a circle – Centre point, diameter and circumference. The situation has not yet changed till date.

The wheel is an unavoidable component in any machine and the circle is an ingredient in human arts which brings about an aesthetic appeal. We also know that the Sphere is the most

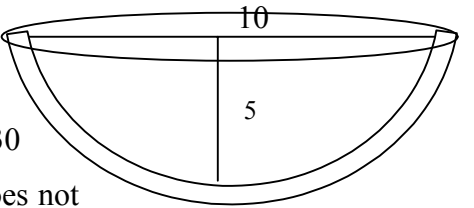
purposes. What length of a tape to be taken to form a wheel of diameter say 2 units? The calculation always failed, little more ore little less. They found that the circumference is slightly more than three times the diameter. If they wanted to find diameter instead it was found slightly less than one third of the circumference.. The knowledge of the exact value of the enhancement in circumference with diameter is important in manufacturing point of view. The problem became more acute as the size of the wheel increased.

Then it all started. To find exact value of this quantity which relates diameter and circumference of circle. Great minds cracked their head – Sulbakaras of India, Archemedes, Aryabhata, Bhaskara, Lebniz, Newton, Lagrange, Ramanujan, Madhava, Aryabhata and a score of computer specialists. In this race the sophistication of mathematical and computational tools brought the Indians to the forefront right from the time of sulbakaras of 1600-600 BC to Madhava of 12^{th} century.

In the modern age Pi present everywhere – in metrology, Basic sciences and engineering and technology This monographs tries to

Oldest mention of Pi is found in the Old Testament

" Also he made a molten sea of ten cubits from brim to brim round in compass and five cubits the height thereof; and a line of thirty cubits and compass it round about." (I Kings VII 23 and 2 in Chronicle IV)



This describes a hemispherical bowl of diameter 10 cubits with circumference of the face 30 cubits. This gives Pi as 3 One does not know how the author of the Old testament arrived at these figures..

An earliest systematic document comes from Babylonian civilization who practiced sexagesimal (base 60) number system. In one of the stone tablets, there is an inscription to measure the circumference of circle with length of six sided inscribed hexagon. The value of Pi obtained is 3.125

.(Converting the hexagesimal to decimal system

We arrive at the following:

Sides of inscribed hexagon = 0.5736 C = (57/60 + 36/60^2) C

1650 BC. Here attempt is made to find volume of a cylinder with diameter 9 khet and height 10 kheit. *It is adviced to subtract 1/9th from 9 (ie, 8) and square it to get area of the circular face which is 64 sqkhet. From this pi is 3.16049*

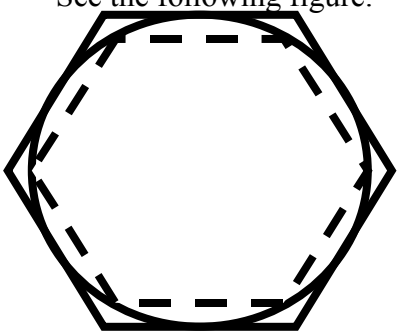
Rich tradition Indian Mathematics gave more insight into the problem from as early as 1600-600 BC in Sulbasutras as descirbes by a number of sulbakars. The knowledge might have been existence from the period of Rigveda which is around 5000BC . This period of Rigveda is fixed from the recent satellite picture showing dried water canal in Rajasthan desert and is attributed to River Saraswathi. Saraswati was a perinneal river during the Rigvedic period reference of which is seen this veda. None of the other three Vedas refers the river Saraswathi. Since the construction of alters were very common during those time, the relevant Mathematical knowledge might have been in existence from that period. Sulbakars might have compiled the then existing knowledge as a written document. One of the objectives of geometry as mentioned above was to construct various shapes of alters to perform rites . The complicated structure of the vedi was

by modern mathematicians.

Geometrical shapes where designed using sulba (threads) and hence the name sulbasutras. Famous sulbakars who made substantial contributions to mathematics are … Sulbasutra gives pi as 3.0888 which was later on corrected to 3.141109 by Dwarakanathan. This shows the problem of introducing correction factors for approximate problems were well appreciated by early Indian Mathematicians.

One can find the upper and lower limit values of Pi as follows.

Inscribe and circumscribe a circle with polygons of n sides. Perimeters of these polygons will be the limits of the value of Pi. See the following figure.



Geometricl method to find upper and lower limits of π which correspond to the perimeters of circumscribed and inscribed regular polygons in a circle .

Archimedes tried to find out the value of Pi by circumscribing and inscribing regular polygons. From 96 sided polygons he was able to find the upper and lower bounds of Pi as 3.140845 < Pi < 3.142857

We come to the AD era when the greatest mind of Indian Mathematics, Aryabhata lived during 476 – 550 AD. This Indian Mathematician and Astronomer in his book Aryabhatiya gives the value of Pi correct to 4 decimal places. As in the following sloka

" Chaturadhikam satamashtagunam dvashashtistatha sahasranaam ayutadvayavishkambahsyasanno vrtaparinaha:"

Meaning , by adding 4 to 100 and multiplied by 8 with this 62000 added and this value is the circumference of a circle of diameter 20000

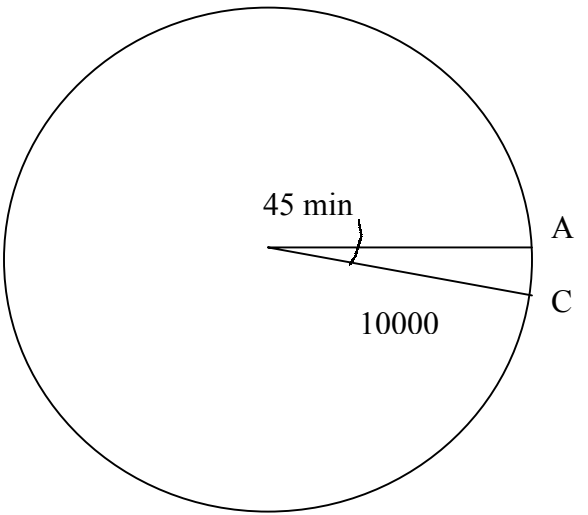
That is pi = 62832/20000 = **3.1416** which is correct to 4 decimal places. He doesnot write how this was arrived at. Bhaskaracarya provides possible method

hence circumference is 150.9 X 480 = 62832

In order to calculate such quantities Aryabhata devised a technique by which a table can be constructed with as accurate as modern trigonometric table . In fact sine is the terminology given to jya when Aryabhatiya was translated into latin from a wrongly translated Arabic text.

Brahmagupta (598 – 660) gave Pi value as square root of 10.

Empirical formula for Pi



Bhaskaracharya (114 – 1185) was one of the greatest mathematicians of his time as well as of the middle age. With him started a new approach to find the value of Pi in which empirical formula were devised to find side of a regular polygon inscribed in a circle.

$S_{2n} = \sqrt{2 - \sqrt{4 - S_n^2}}$ n is the number of sides of the inscribed polygon.

Using 384 sided polygon he found the value of pi as **3.1416** . He called this sukshma value. He gave sthula or gross value as **22/7** which can be used for practical application. *Bhaskaracharya was the first mathematician to recognize pi as ratio*, since he states " when diameter of a circle is multiplied by 3927/1250, the circumference is obtained. "

Pi as infinite series

Before Renaissance in Europe, Indian Mathematicians of middle ages were successful in deriving pi as infinite series. This was known to west only very lately. This marks another type of representation of π.

Talakulaathu Nambudiri in 1431 gave the formula

Madhava, Sankaravarier and Sankara varman) who also made valuable contributions in evaluating the value of π.Some of their formulae were rediscovered by many of the European Mathematicians to whom the credits are given. However, recent realizations made re attribution of some of the discoveries .

For example we have well known Gregory series

$Tan^{-1}x = x- x^3/3 + x^5/5-\ldots$ for $x =<1$ named after the Scotish Mathamatecian James Gregary in 1667. A Kerala Mathematican is credited with the series (translated into modern form) $\theta = \tan\theta - (\tan^3\theta)/3+ (\tan^5\theta)/5 - \ldots$ which is equivalent to Gregory series for inverse tangent. The Madhava series is given in Jyeshtadeva's Yukthibhasha (1550).

With $x=1$ or $\theta = \pi/4$ Madhava- Gregory series is reduced to Euler's series

$\pi/4 = 1-1/3+1/5-1/7+\ldots.$

This series occurs in both Nilakanta's (1445-1555) Tantrasangraha and Puthumana's (1660-1740) Karanapadhathi.

In Tantrasangraha, nilakanta gives a form of Madhava series for PI

As

$\pi d = 4d-4d/3 +4d/5 -\ldots 4d/2n-1 + 4dn/(2n)^2-1$

where d is the diameter of the circle. Madhava gets the value of P with d = $9X10^{11}$, n=25 as 3.14159265359 details of which are found in kriyakaumudi.

Value of pi as infinte seris were evaluated by European Mathematicians during Renaissance period . Among them are Newton, Leibniz and Euler. It was Euler in 1737 *who adopted the symbol Pi and since then it came in general use. Euleer also adopted symbols like e, and i for natural number and square root of –1 respectively*.

Following are some examples of formula for π.

$\pi = 2.4.4.6.6.8.8.10$

*W Brounc*ker$(1620-1684)$ $\frac{4}{\pi} = 1 + \frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}...$

Newton$(1643-1727)$ $\pi = \frac{3\sqrt{3}}{4} + 24[\frac{1}{12} - \frac{1}{5.2^5} - \frac{1}{28.2^7} - ..]$

Leibniz$(1646-1716)$ $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}.....$

J Machin$(1680-1752)$ $\frac{\pi}{4} = 4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239})$

Some of the formulas given by Leonhard Euler (1707 – 1783)are

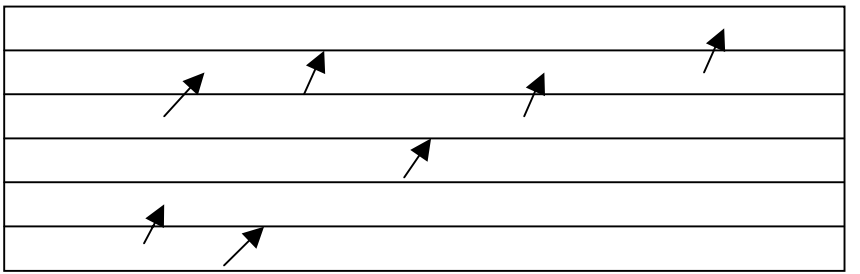
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

$$\frac{2^2\pi^4}{5!3} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + ...$$

$$\frac{76977927.2^{24}\pi^{26}}{27!} = \frac{1}{1^{26}} + \frac{1}{2^{26}} + \frac{1}{3^{26}} + ...$$

Buffon who developed theory of Probability and its applications. One of his problems is as follows:

Draw parallel equidistance lines on a tables separated by distance a. Now take a needle of length l which is less than a and throw it at random on the table. Buffon found that the probabilitiy of needle to cross a line will be $2l/\pi$ a



This is the first documented experimental determination of and provides a very good example of Monte-Carlo technique. Lazzerini found $\pi = 3.1415929$ from 3408 trials. A computer simulation can be used for evaluation of π using this formula.

provided about 4000 formula in a number of notebooks (Called Ramanujan's Notebooks) without any proof. He gave most simplest formula for π as $22\ \pi^4 = 2143$ to more complicated ones as shown below.

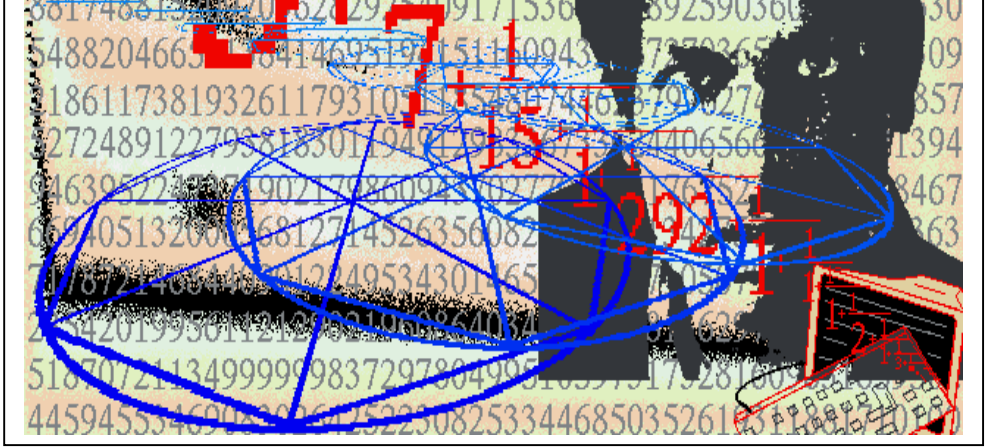
$$(9^2 + \frac{19^2}{22})^{1/4} = 3.14159265262$$

$$\frac{355}{113}(1 - \frac{0.0003}{3533}) = 3.1415926535897943$$

$$\frac{\pi}{2}\log 2 = 1 + \frac{1}{2}.\frac{1}{3^2} + \frac{1.3}{2.4}.\frac{1}{5^2}....$$

$$\frac{105}{\pi^4} = (1 + \frac{1}{2^4})(1 + \frac{1}{3^4})(1 + \frac{1}{5^4}).....$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}.\frac{1}{2}.\frac{1.3}{4^2} + \frac{53883}{99^{10}}.\frac{1.3}{2.4}.\frac{1.3.5.7}{4^2.8^2} + ...$$



In Germany Pi is called Ludolf number .Ludolf Von Ceulen (1539 –1610) in 1610 came up with 35 decimal correct value based on polygon of 2^{62} sides.. According to his wish the value was engraved on his tombstone. In 1755 the French Academy of Science declined to examine any more solutions of π considering this to be a fruitless job. Even then the crazy for the accuracy for π soared high as shown below reaching 808 places in 1947 by

1480	Madhava	11 digits
1593	A V Rooman	15
1610	L V Ceulen	35
1699	A Sharp	71
1706	J Machin	100
1794	G Vega	140
1844	Z Dase	200
1853	W Rutherford	440
1873	W Shank	526
1946	D F Ferguson	620
1947	D F Ferguson and J W Wrench	808

The first computer was made in 1949 called The Electronic Numerical Integrator and Computer (ENIAC) worked for 70 hours to get a result correct to 2037 places using the formula

$Pi = 16 \tan^{-1} 1/5 – 4 \tan^{-1} 1/239$

The race started in 1954 with 3o89 places in 1954 to the record half a million places in 1967 working for 28 hours and 10 min. Details are as given below.

year	Computer	Place	Duration to compute	No. of places
1954	NORC	Virginia	--13min	3089
1957	PEGASUS	London	33hr-	7480
1958	IBM 704	Paris	1hr 40min	10,000
1959	IBM 704	Paris	4hr 18min	16,167
1961	IBM 7090	London	-39min	20,000
1961	IBM 7090	New york	8hr 43min	1,00,265
1966	IBM 7030	Paris	---	2,50,000

1973		France		1million
1981		Japan		2,,
1982		,,		4,,
1982		,,		8,,
1983		,,		10,,
1983		,,		16,,
1985		USA		17,,
1986		,,		29,,
1986		Japan		33,,
1986		,,		67,,
1987		Japan		133 million
1988		,,		201 ,,
2001		Wolfram		200 billion

…

The quantum leap to million places was made possible by the appearance of Super computer. In 1973, Guilloud and Bouyer reached the I million point using a CDC 7600 . In 1986 Baiey of NASA used Ramanuja's last formula to evaluate Pi to 29 million

film for calculation and verification. The print of the figures will take 40266 pages with 5000 figures per page. With five figures per cm the figure will cover 400 km distance from Trivandrum to Kozhikode.

The value Pi now looks like 3.1415926535…….0830068223 in 201,326,000 places. In 2001, Wolfram studied 200 billion digits to know whether there exists any regularities in the occurrence of digits and the result was negative.

Following chart gives pi to 1000 places.

08128 48111 74502 84102 70193 85211 05559 64462 29489 54930
38196 44288 10975 66593 34461 28475 64823 37867 83165 27120
19091 45648 56692 34603 48610 45432 66482 13393 60726 02491
41273 72458 70066 06315 58817 48815 20920 96282 92540 91715
36436 78925 90360 01133 05305 48820 46652 13841 46951 94151
16094 33057 27036 57595 91953 09218 61173 81932 61179 31051
18548 07446 23799 62749 56735 18857 52724 89122 79381 83011
94912 98336 73362 44065 66430 86021 39494 63952 24737 19070
21798 60943 70277 05392 17176 29317 67523 84674 81846 76694
05132 00056 81271 45263 56082 77857 71342 75778 96091 73637
17872 14684 40901 22495 34301 46549 58537 10507 92279 68925
89235 42019 95611 21290 21960 86403 44181 59813 62977 47713
09960 51870 72113 49999 99837 29780 49951 05973 17328 16096
31859 50244 59455 34690 83026 42522 30825 33446 85035 26193
11881 71010 00313 78387 52886 58753 32083 81420 61717 76691
47303 59825 34904 28755 46873 11595 62863 88235 37875 93751
95778 18577 80532 17122 68066 13001 92787 66111 95909 21642
01989...

How it is evolved from a couple of digits to 200 billion digits during a span of 2000 years!!

Pi has also been used to exhibit the memory capacity of human brain. An Indian Mathematician *Rajan Srinivasan Mahadevan in*

40,000 figures of pi. Exceeding Rajan's Record. It is heard that Rajan is preparing to break the record to 1 million. We have not yet heard of whether this feat has been done or not.

Different people memorize following different methods. One of the methods is by using appropriate mnemonics. For example value of Pi with 8 figures can be remembered using the following mnemonics: Each digit corresponds to the number of letters in the words .

Eg: *May I have a large container of coffee* **3 1 4 1 5 9 2 6**

Another one for 15 figures is :

How I want a drink alcoholic of course after the heavy lectures involving

3 1 4 1 5 9 2 6 5 3 5 8 9

quantum mechanics

7 9

1 2 3 4 5 6 7 8 9 0

I J K L M N O P Q R

S T U V W X Y Z [ ] ^ _ ` a

b c e h i j k

zero for pure vowels and no values for semi letters

following quartet gives 32 digits of Pi where each letter is assigned a unique number according to the *katapayadi* system

3 1 4 1 5 9 2 6
gopi bhagya madhu vratha

5 3 5 8 9 7 9 3
srungisodadhi sandhiga

2 3 8 4 6 2 6 4
khala jeevitha khathava

3 3 8 3 2 7 9 5
gala halarasam dhama
which gives the value as

Nature of Pi revealed

Let us now ponder on the character of Pi. There are some numbers which cannot be expressed as fraction of the form p/q and they are called irrational numbers. Eg Sqrt of 2, sqrt of 3 etc. Such numbers have unending nonrecurring decimal expansion.

Now, what type of number is Pi ?

Lambert (1728 – 1777) proved Pi to be irrational and in a rigorous way by Legendre (1752 – 1833) .

If a number can be expressed as an algebraic equation $\Sigma a_n x_n = 0$ it is called algebraic number. For example even though $\sqrt{2}$ is irrational , this number is relates thorugh algebraic equation $x^2 - 2 = 0$ It was George Cantor (1845 – 1918) who showed that all numbers are not algebraic. Euler called them transcendental. J Liouville (1809 – 1882) proved that e cannot satisfy any algebraic equation in 1844 and In 1873, Charles Hermite (1822 – 1905) proved e to be transcendental. In 1882, there was a final settlement of the age old question of squaring a circle as

The irrational number e arises out of binomial expansion $(1+1/n)^n$ as n tends to infinity..

Pi and Complexity

Wolfram in his A New Kind of Science has shown that one can set up all sorts of systems based on numbers in which great complexity can occur.The possibility of such complexity has already been suggested by some some well known facts in elementary mathematics. The sequence of digits appearing in Pi is such an example. Is there a simple regularity law in the digit sequence? Even though definition of Pi is simple, the digit sequence is not simple at all. First 4000 digits both to the base 10 and to the base 2 are shown below. Pictorial representation of the first 20,000 digits is also shown in both bases. No obvious regularities are seen even after going to two hundred billion digits of Pi. Despite the simple definition of Pi, its digit sequence can be considered as random for practical purposes. The complexity of Pi can be visualized when it is expressed in the base 2. First 40000 digits are given in the following page along with the corresponding

Is there a relationship between e and Pi? Computer gave the answer:

$e^6 = \pi^5 + \pi^4$ to an accuracy of 0.000017

The presence of number π is felt everywhere.

As area of a circle (πr^2), surface area of the curved surface of a cone (Pi rl) cylinder (2pirh), volume of sphere ($4/3pir^3$), area of ellipse (piab), area of a cycloid ($3a^2pi$), universal constant in quantum mechanics(h/2pi), Heisenberg's uncertainty principle….it is omnipresent, there is the real existence of π as the circumference of a circle of unit diameter yet immeasurable-transcendental …may be like Brahma (which) is also omnipresent , immeasurable and transcendental . Here we stop when philosophy enters.

And here ends the conducted tour in the realms of history of mathematics to witness the birth and evolution of Pi. The journey never stops . A new formula or new geometrical construction of Pi may appear unexpectedly .

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